Integration by Parts with the DI Method

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Trademarks

1 Introduction

Integration by parts is a technique for integrating products of two functions. Although the technique is fairly straightforward, it can be tedious to perform by hand, requiring both differentiation and integration. The DI, or tabular, method is a way to organize the computations involved in performing integration by parts. The functionality of TI-Nspire's Lists and Spreadsheets application can be used to automate the DI Method and to simplify integrating products of two functions.

This article describes integration by parts in general, then describes how to perform integration by parts with the DI Method. Next, the article describes how to configure a TI-Nspire spreadsheet to automate the DI Method. Finally, examples are presented that show how to use the spreadsheet to integrate products of functions for which integration by parts can be used.

The TI-Nspire document that accompanies this article contains the examples described in this article, and requires the CAS version of TI-Nspire.

2 Integration by Parts

2.1 Derivation

The formula for integration by parts is derived from the product rule for differentiating the product of two functions: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. Solving this equation for $f(x)g'(x)$ then integrating both sides of the equation yields the formula for integration by parts:

$$f(x)g'(x) = \frac{d}{dx}[f(x)g(x)] - f'(x)g(x) \tag{1}$$

$$\int f(x)g'(x)dx = \int \frac{d}{dx}[f(x)g(x)] - \int f'(x)g(x)dx \tag{2}$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx \tag{3}$$

Equation (3) is the formula for integrating by parts. The equation is simplified using substitution, resulting in the standard textbook formula for integration by parts:

Let $u = f(x)$, then $du = f'(x)dx$
Let $dv = g'(x)dx$, then $v = \int g'(x)dx = g(x)$

With these substitutions, the formula for integration by parts simplifies to

$$\int udv = uv - \int vdu$$

Notice that the solution to $\int udv$ contains an integral, $\int vdu$. Often, solving the integral $\int vdu$ will also require integration by parts. Because of this, solving $\int udv$ may require repeated use of the technique. Occasionally, $\int vdu$ will be a multiple of $\int udv$ which requires algebraically manipulating the result to arrive at a solution.

2.2 Examples

The following examples illustrate how to evaluate indefinite integrals using integration by parts.

Example 2.2.1. *Evaluate* $\int x \sin(x) dx$.

Solution:

Let $u = x$, then $du = dx$
Let $dv = \sin(x)dx$, then $v = \int \sin(x)dx = -\cos(x)$

Note: substituting u for x and equating $\int x \sin(x)dx$ with $\int udv$ implies $dv = \sin(x)dx$.

Integration by parts using these substitutions is

$$\begin{aligned}\int x \sin(x)dx &= -x\cos(x) - \int (-\cos(x))dx \\ &= -x\cos(x) + \int \cos(x)dx \\ &= \sin(x) - x\cos(x)\end{aligned}$$

Adding the required constant of integration, the solution is

$$\int x \sin(x)dx = \sin(x) - x\cos(x) + C$$

Example 2.2.2. *Evaluate* $\int x^2 \sin(x) dx$.

Solution:

Let $u = x^2$, then $du = 2xdx$
Let $dv = \sin(x)dx$, then $v = \int \sin(x)dx = -\cos(x)$

Integration by parts using these substitutions is

$$\begin{aligned}\int x^2 \sin(x)dx &= -x^2\cos(x) - \int (-\cos(x))2xdx \\ &= -x^2\cos(x) + 2\int x\cos(x)dx\end{aligned}$$

At this point, the right-hand side of the equation contains an integral that also requires integration by parts: $\int x\cos(x)dx$. To evaluate this integral,

Let $u = x$, then $du = dx$
Let $dv = \cos(x)dx$, then $v = \sin(x)$

With these substitutions and continuing the process,

$$\begin{aligned}
\int x^2 \sin(x) dx &= -x^2 \cos(x) - \int (-\cos(x)) 2x dx \\
&= -x^2 \cos(x) + 2 \int x \cos(x) dx \\
&= -x^2 \cos(x) + 2[x \sin(x) - \int \sin(x) dx] \\
&= -x^2 \cos(x) + 2[x \sin(x) - (-\cos(x))] \\
&= -x^2 \cos(x) + 2[x \sin(x) + \cos(x)] \\
&= -x^2 \cos(x) + 2\cos(x) + 2x \sin(x)
\end{aligned}$$

Adding the constant of integration, the solution is

$$\int x^2 \sin(x) dx = -x^2 \cos(x) + 2\cos(x) + 2x \sin(x) + C$$

Example 2.2.3. *Evaluate* $\int \sin(x)\mathbf{e}^x dx$*.*

Solution:

Let $u = \sin(x)$, then $du = \cos(x)dx$
Let $dv = \mathbf{e}^x dx$, then $v = \mathbf{e}^x$

Integration by parts using these substitutions is

$$\int \sin(x)\mathbf{e}^x dx = \sin(x)\mathbf{e}^x - \int \mathbf{e}^x \cos(x)) dx$$

The right-hand side of the equation contains an integral requiring integration by parts: $\int \mathbf{e}^x \cos(x))dx$. To evaluate this integral,

Let $u = \cos(x)$, then $du = -\sin(x)dx$
Let $dv = \mathbf{e}^x dx$, then $v = \mathbf{e}^x$

Substituting these values into the equation and continuing

$$\begin{aligned}
\int \sin(x)\mathbf{e}^x dx &= \sin(x)\mathbf{e}^x - \int \mathbf{e}^x \cos(x)) dx \\
&= \sin(x)\mathbf{e}^x - [\cos(x)\mathbf{e}^x - \int \mathbf{e}^x (-\sin(x) dx] \\
&= \sin(x)\mathbf{e}^x - [\cos(x)\mathbf{e}^x + \int \mathbf{e}^x \sin(x) dx] \\
&= \sin(x)\mathbf{e}^x - \cos(x)\mathbf{e}^x - \int \mathbf{e}^x \sin(x) dx
\end{aligned}$$

After this evaluation, the right-hand side of the equation contains an integral that is equal to the integral being evaluated: $\int \mathbf{e}^x \sin(x)dx = \int \sin(x)\mathbf{e}^x dx$. Thus continuing to integrate by parts would never end. However, the last equation can be solved algebraically by adding $\int \mathbf{e}^x \sin(x)dx$ to both sides of the equation:

$$\begin{aligned}
\int \sin(x)\mathbf{e}^x dx + \int \mathbf{e}^x \sin(x)dx &= \sin(x)\mathbf{e}^x - \cos(x)\mathbf{e}^x - \int \mathbf{e}^x \sin(x)dx + \int \mathbf{e}^x \sin(x)dx \\
2\int \sin(x)\mathbf{e}^x dx &= \sin(x)\mathbf{e}^x - \cos(x)\mathbf{e}^x \\
\int \sin(x)\mathbf{e}^x dx &= \frac{1}{2}[\sin(x)\mathbf{e}^x - \cos(x)\mathbf{e}^x]
\end{aligned}$$

After rearranging the equation and adding the integration constant, the solution is

$$\int \sin(x)\mathbf{e}^x dx = \frac{\mathbf{e}^x \sin(x)}{2} - \frac{\mathbf{e}^x \cos(x)}{2} + C$$

3 Considerations

When confronted with an integral that is the product of two functions, the first thing to consider is whether or not integration by parts is the correct approach to evaluate the integral. For many such integrals, *substitution* is the technique to use[1]. As an example, the integral $\int x^2 \sin(x^3)dx$ which appears to be a good candidate for integration by parts can be easily evaluated using the substitution rule.

Once the decision to use integration by parts is made, several issues need to be considered before applying the technique:

1. Which of the two functions $f(x)$ and $g(x)$ to choose for u and dv.
2. When to stop the process if repeated application of integration by parts is required.

3.1 Choosing u and dv

The choice of which function to use for u and which to use for dv in the equation $\int u dv = \int f(x)g'(x)dx$ can make integration by parts an easy or difficult task. For some integrals, the wrong choice can render the task difficult or impossible to complete. A general rule for the choice is to choose u as the function that is simpler when differentiated or choose dv as the function that is simpler when integrated [4]. A more specific rule of thumb for the choice is exemplified by the **LIATE** rule: choose u as the function that appears first in the following list and choose dv as the function that is last in the list[4]:

[1] The substitution rule is appropriate for integrals of the form $\int f(g(x))g'(x)dx$

Abbreviation	Function Type	Examples
L	logarithmic	$\ln x, \log_{10}(x)$
I	inverse trigonometric	$\arcsin(x), \arctan x$
A	algebraic	$x^2, 5x^2+x+1$
T	trigonometric	$\sin(x), \cos(x)$
E	exponential	$\mathbf{e}^x, 8^x$

A few examples of choices for u and dv using the **LIATE** rule are

Integral	u	dv
$\int x\ln(x)dx$	$\ln(x)$	xdx
$\int \mathbf{e}^{2x}\cos(x)$	$\cos(x)$	$\mathbf{e}^{2x}dx$
$\int x^2\sin(x)dx$	x^2	$\sin(x)dx$
$\int \arcsin(x)dx$	$\arcsin(x)$	dx

3.2 Stopping the Process

For many integrals, integration by parts must be repeated several times before a solution is found. In this case, knowing when to stop the process is important. There are three conditions when the process should be stopped:

1. When the integral $\int vdu$ can be integrated without applying integration by parts.
2. When the integral $\int vdu$ is equal to or a multiple of the integral being evaluated. When this happens, the solution is found by solving the resulting equation algebraically. See Example 2.2.3, the example of integrating $\int \sin(x)\mathbf{e}^x dx$, to see how the equation is solved.
3. When the derivative and/or integral becomes more complex with each iteration. In this case, either a bad choice for u or dv was made, or integration by parts may not be the appropriate method for evaluating the integral.

4 Integration by Parts for Definite Integrals

Integration by parts for definite integrals is defined as follows

$$\begin{aligned}\int_a^b udv &= (uv)|_a^b - \int_a^b vdu \\ &= u(b)v(b) - u(a)v(a) - \int_a^b vdu\end{aligned}$$

The only difference between evaluating definite and indefinite integrals is that for definite integrals, the evaluation includes calculating the value of the integral using the lower and upper bounds of the definite integral. The calculation can be performed either during each step of the process with the intermediate result, or at the end of the process with the final result. When repeated application of integration by parts is required, calculating $(uv)|_a^b$ at each step of the process becomes cluttered and error prone. Because of this, deferring the calculation of the value of the definite integral until the final result (the antiderivative) has been found is simpler and less subject to error.

Example 2.2.3, demonstrating how integrating $\int \sin(x)\mathbf{e}^x dx$ requires several repetitions of integration by parts, along with algebraically solving for the solution, provides a good example of deferring evaluation of a definite integral until the antiderivative is found.

Example 4.0.1. *Evaluate* $\int_0^\pi \sin(x)\mathbf{e}^x dx$.

Solution:

In Example 2.2.3, the antiderivative of $\int \sin(x)\mathbf{e}^x dx$ (without the constant of integration) was found to be

$$\frac{\mathbf{e}^x \sin(x)}{2} - \frac{\mathbf{e}^x \cos(x)}{2}$$

The value of the definite integral is calculated using the Fundamental Theorem of Integral Calculus[2]:

$$\begin{aligned}
\int_0^\pi \sin(x)\mathbf{e}^x dx &= \left[\frac{\mathbf{e}^x \sin(x)}{2} - \frac{\mathbf{e}^x \cos(x)}{2}\right]\Bigg|_0^\pi \\
&= \left[\frac{\mathbf{e}^\pi \sin(\pi)}{2} - \frac{\mathbf{e}^\pi \cos(\pi)}{2}\right] - \left[\frac{\mathbf{e}^0 \sin(0)}{2} - \frac{\mathbf{e}^0 \cos(0)}{2}\right] \\
&= \frac{\mathbf{e}^\pi}{2} - \left(-\frac{1}{2}\right) \\
&= \frac{\mathbf{e}^\pi}{2} + \frac{1}{2}
\end{aligned}$$

5 The DI Method

The **DI**, or **Tabular** Method organizes each step of integration by parts in a table. An example demonstrating how the table is set up and how the steps in the process are captured in rows of the table follows.

Example 5.0.1. *Find the antiderivative of* $\int x^2 \sin(x)dx$ *using the DI Method.*

Initializing the Table: The function chosen as u is placed in one column and the function chosen as dv is placed in an adjacent column, and another column containing

[2] $\int_a^b f(x)dx = F(b) - F(a)$ where $F(x)$ is an antiderivative of $f(x)$.

the multiplier for the sign of the products uv and vdu is placed in a third column. Based on the **LIATE** rule, for this integral, x^2 is chosen as u and $\sin(x)$ is chosen as dv. The initial table with these values is

index	**u**	**dv**	**sign**
0	x^2	$\sin(x)$	

After the table is initialized, each step in the process is performed by adding a row to the table containing the following entries: the derivative of the preceding entry in the u column, the integral of the preceding entry in the dv column, and an alternating sign in the third column, beginning with a $+$ sign.

The Table after Step 1:

index	**u**	**dv**	**sign**
0	x^2	$\sin(x)$	
1	$2x$	$-\cos(x)$	$+$

After a new row is added to the table, all the components for integration by parts is in the table: the value for u is in row $index-1$, the value for du is in row $index$, the value for v is in row $index$, and the multiplier for the sign of uv and vdu is in row $index$. For this example, after step 1, $u = x^2, v = -\cos(x), uv = -x^2\cos(x), du = 2x$ and $vdu = -\cos(x)2x$. Notice that uv is calculated by multiplying **diagonal** entries in the u column and the dv column: the $u(0)$ entry times the $dv(1)$ entry. The intermediate value for the integral is

$$-x^2\cos(x) + 2\int \cos(x)xdx$$

The process of adding a row and calculating values continues as long as the integral $\int vdu$ requires integration by parts or until it is equal to or a multiple of the original integral.

The Table after Step 2:

index	**u**	**dv**	**sign**
0	x^2	$\sin(x)$	
1	$2x$	$-\cos(x)$	$+$
2	2	$-\sin(x)$	$-$

After step 2, the values for the intermediate calculation are $u = 2x, v = -\sin(x), du = 2$, the multiplier for uv and vdu is -1, so $uv = -(-2x\sin(x)) = 2x\sin(x), vdu = -(-2\sin(x)) = 2\sin(x)$ and the intermediate value for the integral is

$$-x^2\cos(x) + 2x\sin(x) - 2\int \sin(x)dx$$

Since $\int \sin(x)dx$ can be easily integrated, the process could be stopped at this point. However, since polynomials can be differentiated until their derivative equals 0, one more step can be performed, yielding the final result.

The Table after Step 3:

index	**u**	**dv**	**sign**
0	x^2	$\sin(x)$	
1	$2x$	$-\cos(x)$	$+$
2	2	$-\sin(x)$	$-$
3	0	$\cos(x)$	$+$

After step 3, the values for the calculation are $u = 2, v = \cos(x), du = 0$, the multiplier for the sign of uv and vdu is $+1$, so $uv = 2\cos(x), vdu = 0$ and the final value for the integral is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) - \int 0dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x)$$

6 Additional Information

For theoretical justification of the DI Method along with examples of applying the method, see the reference article ”The Tabular Method for Repeated Integration by Parts”[1]. The web page ”Integration by Parts”[4] contains excellent descriptions of integration by parts, including descriptions of the LIATE Rule and the Tabular Method, and provides usage examples. The book, *Calculus II for Dummies*[5], describes the DI Method in detail and presents examples of using the method to integrate products composed of logarithmic, inverse trigonometric, algebraic, and trigonometric functions.

7 The DI Method with TI-Nspire

TI-Nspire’s **Lists and Spreadsheets** application can be configured to perform integration by parts with the DI Method. This section shows how to configure a spreadsheet to perform each step in the process and automatically carry out all the required calculations.

7.1 Configuring the Spreadsheet

The following images and explanations show how to configure a TI-Nspire spreadsheet to find the antiderivative of an indefinite integral with the DI Method. The integral $\int(3x+5)\cos(\frac{x}{4})dx$ from ”Calculus II”[2] is used as an example. For this integral, the polynomial $3x+5$ is chosen as the value for u and $\cos(\frac{x}{4})dx$ is chosen as the value for dv.

To begin, create a new document and add a spreadsheet to the document (optionally, use a split page containing a spreadsheet and a Calculator page or a Notes page). Add labels to columns A thru F of the spreadsheet. The values in each column in a spreadsheet are stored in a list and the column labels are the names of the lists. The labels are used to refer to the lists in a Calculator or Notes page. The contents of the columns are:

Column	**Label**	**Contents**
A	u	u and derivatives of u (du)
B	dv	dv and integrals of dv (v)
C	pm	Multiplier for sign of u times v and v times du
D	uv	Product of u, v and pm
E	vdu	Product of v, du and pm
F	sum_uv	Sum of uv

Enter the values in row 1 of the spreadsheet as shown in Figure 1. The initial configuration of the spreadsheet is

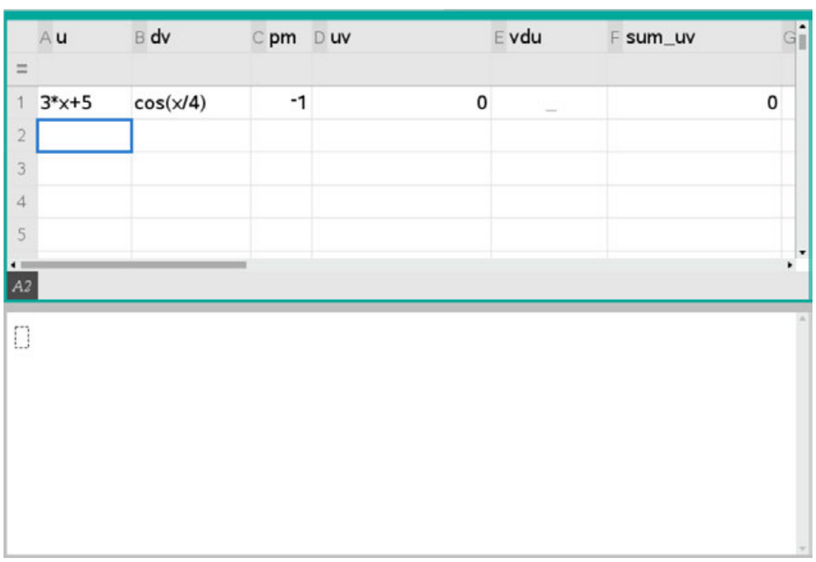
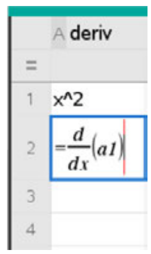
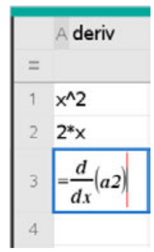


Figure 1: Initial Spreadsheet Configuration

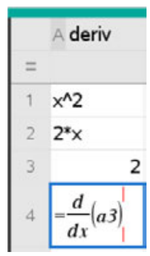
Completing the configuration of the spreadsheet requires defining the formulas to perform the integration by parts calculations. These calculations consist of differentiating u, integrating dv, finding the products uv and vdu, and adding the products uv. The formulas for these calculations are entered by typing the expressions for the formulas in cells in row 2 of the spreadsheet. The formulas are defined using *relative cell references*. When a relative cell reference is copied, the cell reference is automatically updated[3]. Figure 2 shows how a formula with a relative cell reference can be used to find all the derivatives of a function in a spreadsheet column.



(a) References Cell A1



(b) References Cell A2



(c) References Cell A3

(d) Final Result

Figure 2: Derivatives Using Relative Cell References

As shown in Figure 2a, the function to differentiate is entered in cell A1 and the formula for the derivative of the function is entered in cell A2, referencing the contents of cell A1 (the function to differentiate). After cell A2 is entered, the derivative of the function in cell A1 is calculated. Next, cell A2 is selected and copied to cell A3. The reference to A1 is automatically updated to reference cell A2 and the derivative of the function in cell A2 is calculated. Finally, selecting and copying cell A3 to cell A4 results in the reference to A2 being automatically updated to reference cell A3 and the derivative of the function in cell A3 is calculated. Note that the formula for finding the derivative was only explicitly entered once, in cell A2. The spreadsheet automatically updated the relative cell reference and calculated the next derivative when a cell was copied.

The following images and descriptions illustrate how to complete configuring the spreadsheet to perform integration by parts using relative cell references.

Differentiating u:

The derivatives of $u, 3x+5$, are calculated in Column A, and the formula for calculating the derivatives using relative referencing is placed in cell A2. The formula is entered by placing the cursor in cell A2, then double-clicking the derivative template in the **Math Templates** panel. Figure 3 depicts the spreadsheet after the formula is entered.

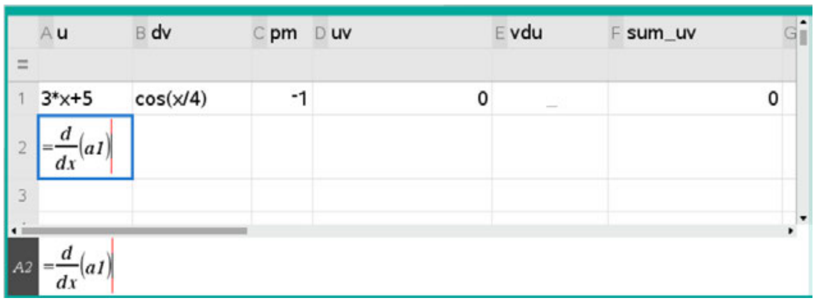


Figure 3: Differentiating **u**

Integrating dv:

The integrals of $dv, \cos(\frac{x}{4})$, are calculated in Column B, and the formula for calculating the integrals using relative referencing is placed in cell B2. The formula is entered by placing the cursor in cell B2, then double-clicking the indefinite integral template in the **Math Templates** panel. Figure 4 depicts the spreadsheet after the formula is entered.

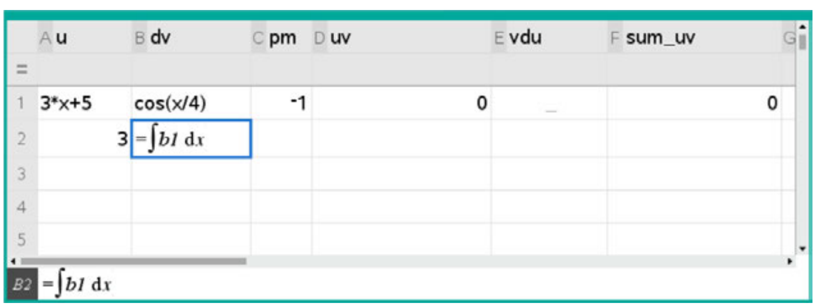


Figure 4: Integrating **dv**

Specifying the multiplier for the sign of uv and vdu:

The sign of the products uv and vdu alternates between plus and minus when repeated integration by parts is required. The sign is calculated in Column C, with initialization value placed in cell C1 and the formula for the sign entered in cell C2 using relative referencing as $-1 * C1$. The entry is either typed directly in cell C2 or typed in the entry line at the bottom of the spreadsheet. Figure 5 shows the spreadsheet after the formula for the sign is entered.

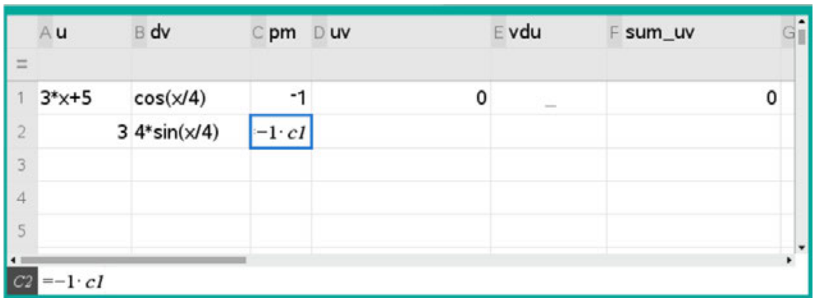


Figure 5: Specifying the sign of **uv**

Finding the product uv:

The initial value of *u* is in cell A1, the initial value of *v* is in cell B2, and the initial multiplier for the sign of *uv* is in cell C2. The relative cell reference for the initial value of *uv* is A1*B2*C2 which is the formula placed in cell D2 as shown in Figure 6.

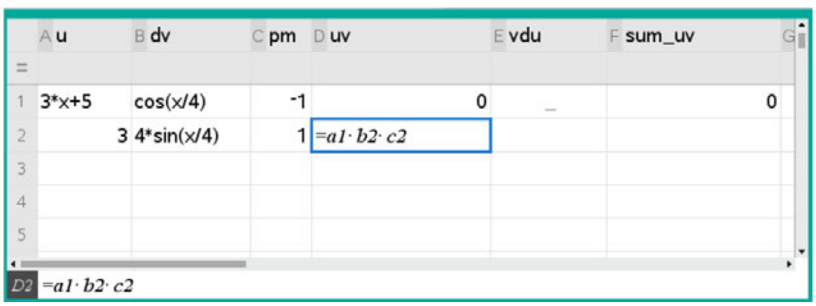
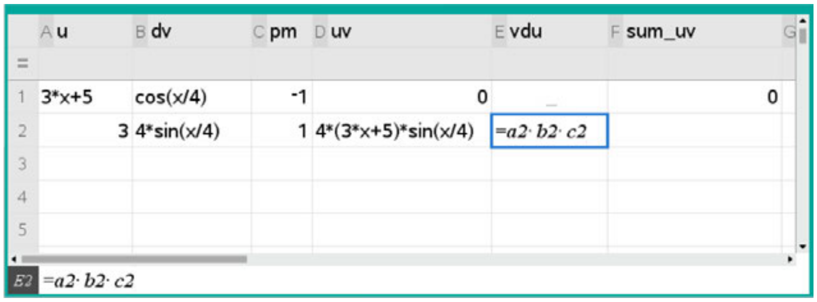


Figure 6: Finding the product **uv**

Finding the product vdu:

The initial value of *du* is in cell A2, the initial value of *v* is in cell B2, and the initial multiplier for the sign of *vdu* is in cell C2. The relative cell reference for the initial value of the integrand *vdu* is A2*B2*C2 which is the formula placed in cell E2 as shown in Figure 7.



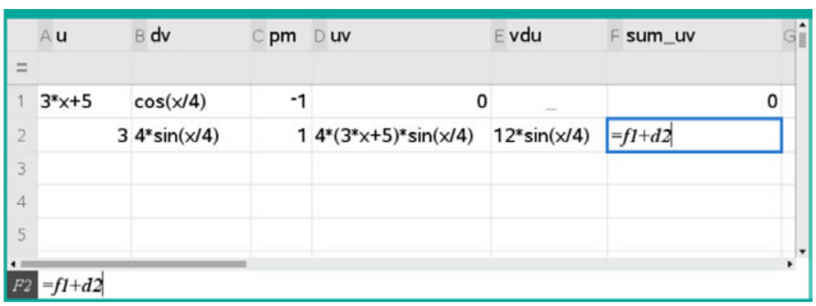
	A u	B dv	C pm	D uv	E vdu	F sum_uv
=						
1	3*x+5	cos(x/4)	-1	0	_	0
2	3	4*sin(x/4)	1	4*(3*x+5)*sin(x/4)	=a2· b2· c2	
3						
4						
5						

E2 =a2· b2· c2

Figure 7: Finding the product **vdu**

Summing the products uv:

The cumulative sum of the product of *u* and *v* is calculated with relative cell references in Column F. Cell F1 contains the initial sum 0 and cell D2 contains the initial value for *uv*. The formula for accumulating the sum with relative cell references is F1+D2 which is the formula entered in cell F2 as shown in Figure 8.

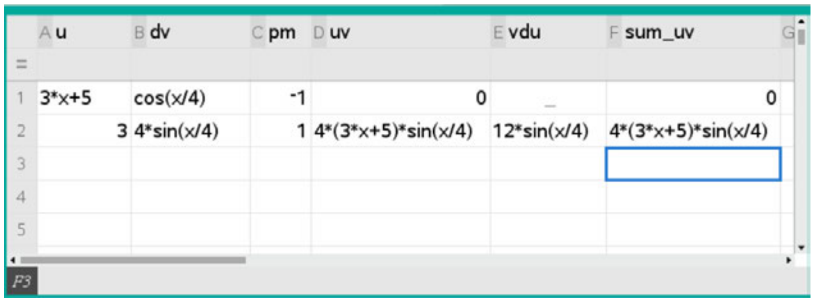


	A u	B dv	C pm	D uv	E vdu	F sum_uv
=						
1	3*x+5	cos(x/4)	-1	0	_	0
2	3	4*sin(x/4)	1	4*(3*x+5)*sin(x/4)	12*sin(x/4)	=f1+d2
3						
4						
5						

F2 =f1+d2

Figure 8: Summing the products **uv**

After the entry for cell F2 is entered, the configuration for the spreadsheet is complete and the initial calculations are finished. The spreadsheet appears as displayed in Figure 9.



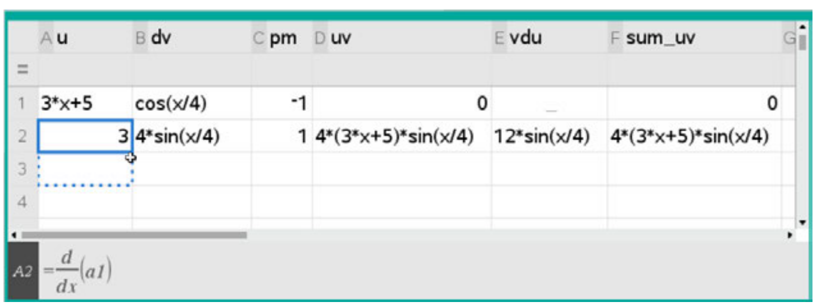
	A u	B dv	C pm	D uv	E vdu	F sum_uv
=						
1	3*x+5	cos(x/4)	-1	0	_	0
2	3	4*sin(x/4)	1	4*(3*x+5)*sin(x/4)	12*sin(x/4)	4*(3*x+5)*sin(x/4)
3						
4						
5						

F3

Figure 9: Configured Worksheet with Initial Calculations

7.2 Evaluating an Integral with the Spreadsheet

After completely configuring the spreadsheet, evaluating the indefinite integral is accomplished by copying each cell to the next row in left-to-right order: cell A2 is copied to cell A3, cell B2 is copied to cell B3, ..., cell F2 is copied to cell F3. As each cell is copied, the relative references are automatically updated and new values for the cells are automatically calculated. This process is displayed in the following figures.

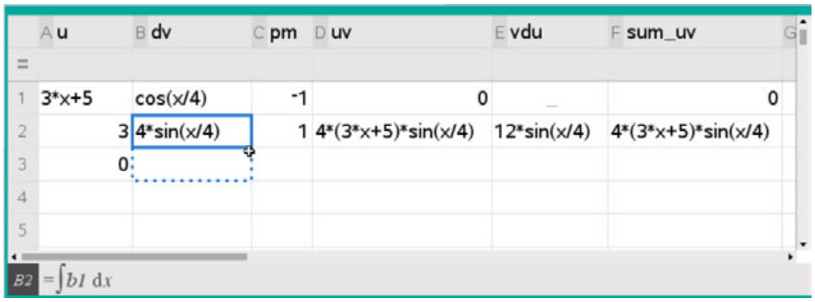


	A u	B dv	C pm	D uv	E vdu	F sum_uv
=						
1	3*x+5	cos(x/4)	-1	0	_	0
2	3	4*sin(x/4)	1	4*(3*x+5)*sin(x/4)	12*sin(x/4)	4*(3*x+5)*sin(x/4)
3						
4						

A2 $=\frac{d}{dx}(a1)$

Figure 10: Copying Cell A2 to Cell A3

As shown in Figure 10, the simplest and most efficient way to copy a cell to the following row is by moving the cursor to the bottom, right side of the cell, selecting the cell with the mouse, then dragging the bottom of the cell into the next row. The cursor changes to a **+** when the cell is selected. When the mouse button is released, the contents of the copied cell are added to the target cell, the relative references are automatically updated, and the new value is added to the cell as shown in Figure 11.

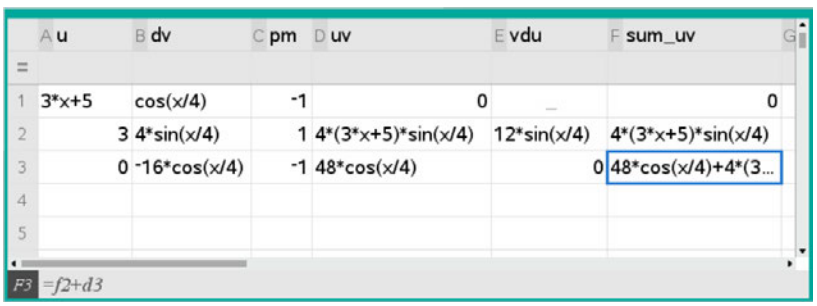


	A u	B dv	C pm	D uv	E vdu	F sum_uv
=						
1	3*x+5	cos(x/4)	-1	0	_	0
2	3	4*sin(x/4)	1	4*(3*x+5)*sin(x/4)	12*sin(x/4)	4*(3*x+5)*sin(x/4)
3	0					
4						
5						

B2 $=\int b1\,dx$

Figure 11: Copying Cell B2 to Cell B3

The same technique for copying cell A2 to cell A3 is applied to cells B2 thru F2, in left-to-right order. After copying cells B2 to B3, C2 to C3, D2 to D3, E2 to E3, and F2 to F3, the spreadsheet appears as shown in Figure 12.



	A u	B dv	C pm	D uv	E vdu	F sum_uv
=						
1	3*x+5	cos(x/4)	-1	0	_	0
2	3	4*sin(x/4)	1	4*(3*x+5)*sin(x/4)	12*sin(x/4)	4*(3*x+5)*sin(x/4)
3	0	-16*cos(x/4)	-1	48*cos(x/4)	0	48*cos(x/4)+4*(3...
4						
5						

F3 $=f2+d3$

Figure 12: The Spreadsheet after Copying Cell F2 to Cell F3

For this example, both du and vdu are now equal to 0 and integration by parts is complete. Since $\int vdu$ is equal to 0, the value of the indefinite integral (without the constant of integration) is equal to the cumulative sum of uv, which is the value displayed in cell F3. The result is verified by calculating the integral in a Calculator page as shown in Figure 13.

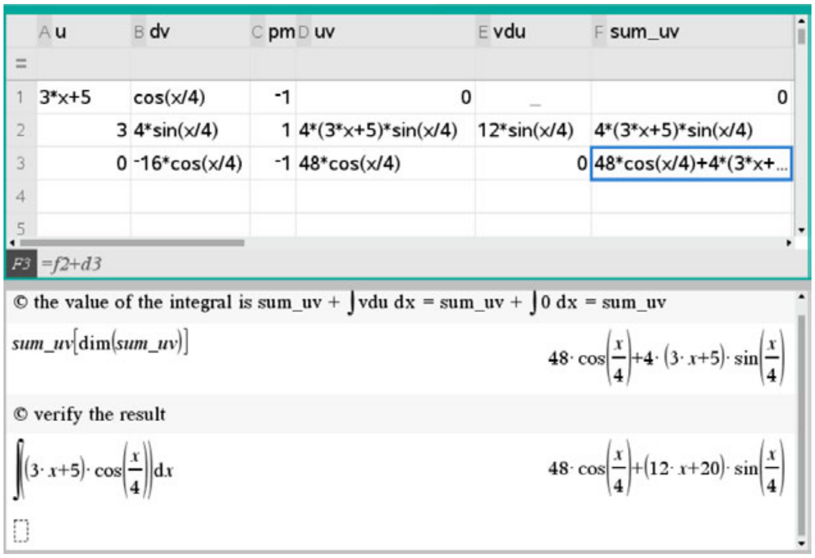


Figure 13: Verifying the Final Result in a Calculator Page

7.3 Notes

1. Copying cells in left-to-right column order (A,B,C,D,E,F) is necessary because the formula in cell D2 references cells A1, B2, and C2, the formula in cell E2 references cells A2, B2, and C2, and the formula in cell F2 references cell D2.

2. Although copying and pasting cells using the **copy** and **paste** menu items (or the keyboard shortcuts **ctrl-c, ctrl-v**) is supported by the **Lists & Spreadsheet** application, copying cells D2 and E2 using these techniques fails even though copying by selecting and dragging succeeds.

3. For some integrals, repeated application of integration by parts results in the integral $\int vdu$ being equal to or a multiple of the integral being evaluated. The value vdu is calculated in column D of the worksheet. Thus, after a cell in column D is copied and its value displayed, the value should be examined to determine if it is equal to or a multiple of the integral being evaluated. If this is the case, then the last results should be used to algebraically find the value of the integral. See Example 2.2.3 and Example 8.0.2 for examples of how to find the value of such integrals.

4. Some integrals will require evaluating multiple integrals to find the value of the original integral. Example 8.0.5 and Example 8.0.6 show how to solve this type of integral.

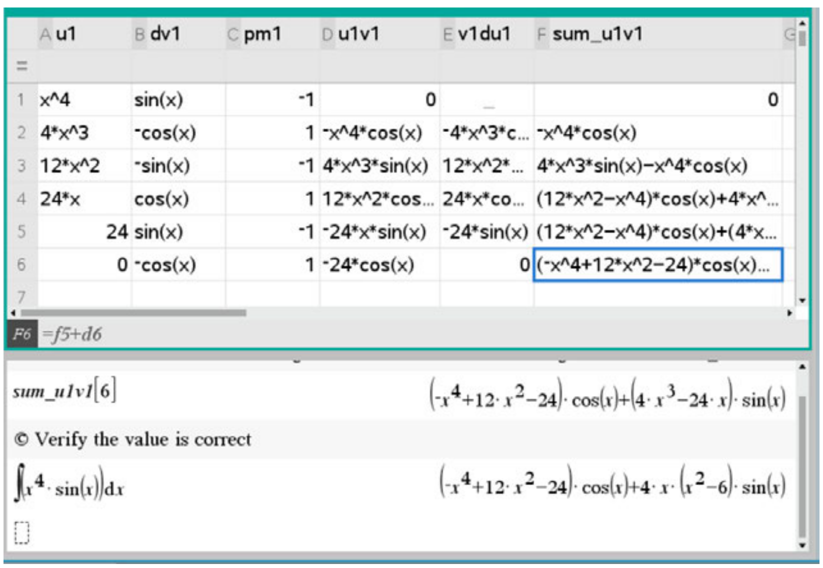
8 Examples

The following examples show how to use the LIATE Rule and the DI method in a TI-Nspire spreadsheet to evaluate various integrals.

Example 8.0.1. *Evaluate* $\int x^4 \sin(x) dx$.

Discussion:

Choose x^4 as u and $\sin(x)$ as dv and configure a TI-Nspire spreadsheet as described above, then copy spreadsheet cells until the derivative of u equals zero. The value of the indefinite integral (without the constant of integration) is then equal to the sum of uv. Figure 14 shows the final result of the process.



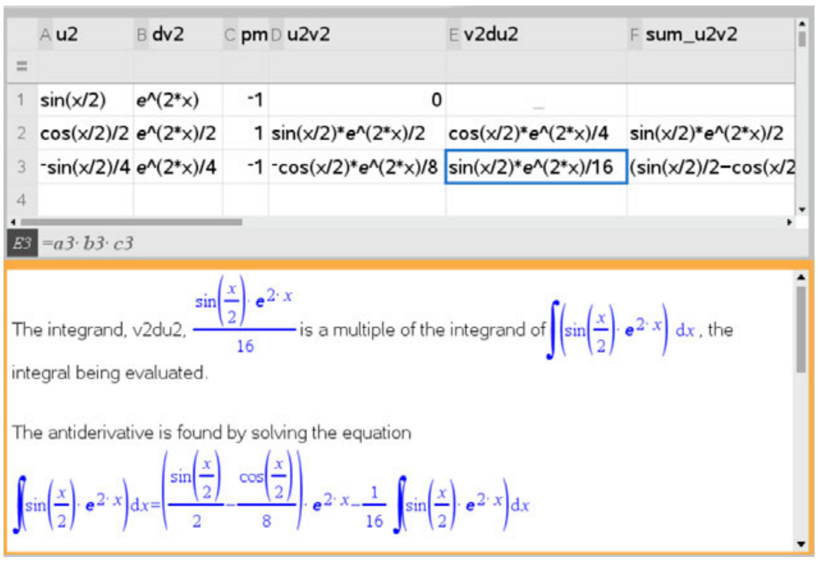
	A u1	B dv1	C pm1	D u1v1	E v1du1	F sum_u1v1
=						
1	x^4	sin(x)	-1	0	_	0
2	4*x^3	-cos(x)	1	-x^4*cos(x)	-4*x^3*c...	-x^4*cos(x)
3	12*x^2	-sin(x)	-1	4*x^3*sin(x)	12*x^2*...	4*x^3*sin(x)−x^4*cos(x)
4	24*x	cos(x)	1	12*x^2*cos...	24*x*co...	(12*x^2−x^4)*cos(x)+4*x^...
5	24	sin(x)	-1	-24*x*sin(x)	-24*sin(x)	(12*x^2−x^4)*cos(x)+(4*x...
6	0	-cos(x)	1	-24*cos(x)	0	(-x^4+12*x^2−24)*cos(x)...
7						

Figure 14: Evaluation of $\int \mathbf{x^4} \sin(\mathbf{x}) \mathbf{dx}$ with the DI Method

Example 8.0.2. *Evaluate* $\int \sin(\frac{x}{2})\mathbf{e}^{2x}dx$.

Discussion:

Choose $\sin(\frac{x}{2})$ as u and $\mathbf{e}^{2x}$ as dv and configure a TI-Nspire spreadsheet, then copy spreadsheet cells until the integrand vdu equals a multiple of the integrand of $\sin(\frac{x}{2})\mathbf{e}^{2x}$. The value of the indefinite integral (without the constant of integration) is then calculated algebraically. Figure 15 shows the final result of the process.



	A u2	B dv2	C pm	D u2v2	E v2du2	F sum_u2v2
=						
1	sin(x/2)	e^(2*x)	-1	0		
2	cos(x/2)/2	e^(2*x)/2	1	sin(x/2)*e^(2*x)/2	cos(x/2)*e^(2*x)/4	sin(x/2)*e^(2*x)/2
3	-sin(x/2)/4	e^(2*x)/4	-1	-cos(x/2)*e^(2*x)/8	sin(x/2)*e^(2*x)/16	(sin(x/2)/2−cos(x/2
4						

E3 $=a3\cdot b3\cdot c3$

The integrand, v2du2, $\frac{\sin\left(\frac{x}{2}\right)\cdot e^{2\cdot x}}{16}$ is a multiple of the integrand of $\int\left(\sin\left(\frac{x}{2}\right)\cdot e^{2\cdot x}\right)dx$, the integral being evaluated.

The antiderivative is found by solving the equation

$$\int\left(\sin\left(\frac{x}{2}\right)\cdot e^{2\cdot x}\right)dx=\left(\frac{\sin\left(\frac{x}{2}\right)}{2}-\frac{\cos\left(\frac{x}{2}\right)}{8}\right)\cdot e^{2\cdot x}-\frac{1}{16}\int\left(\sin\left(\frac{x}{2}\right)\cdot e^{2\cdot x}\right)dx$$

Figure 15: Evaluation of $\int \sin(\frac{x}{2})\mathbf{e}^{2x}dx$ with the DI Method

Example 8.0.3. *Evaluate* $\int x \ln(x^2) dx$.

Discussion:

Choose $\ln(x^2)$ as u and x as dv and configure a TI-Nspire spreadsheet. When configuration is complete, the value of vdu is just x, which is easy to integrate. The value of the indefinite integral (without the constant of integration) is then simply calculated as $uv - \int vdu$. Figure 16 shows the final result of the process.

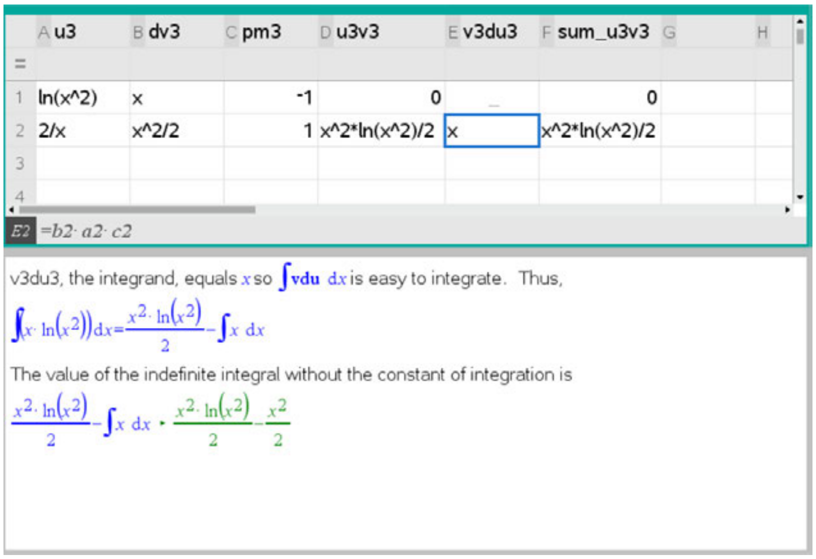


Figure 16: Evaluation of $\int x \ln(x^2) dx$ with the DI Method

Example 8.0.4. *Evaluate* $\int \arctan(x)dx$.

Discussion:

At first glance, the integrand does not appear to be a product and the integral does not seem to be a candidate for integration by parts. However, the integrand can be written as the product of $\arctan(x)$ and $1dx = dx$. Choose $\arctan(x)$ as u and 1 as dv and configure a TI-Nspire spreadsheet. When configuration is complete, the value of vdu is $\frac{x}{x^2+1}$, which can be integrated using substitution. The value of the indefinite integral (without the constant of integration) is then simply calculated as $uv - \int vdu$. Figure 17 shows the final result of the process.

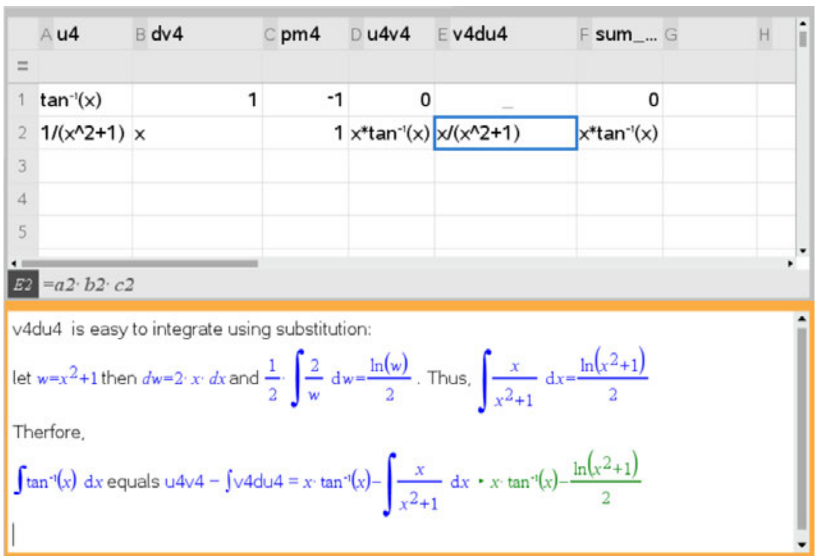
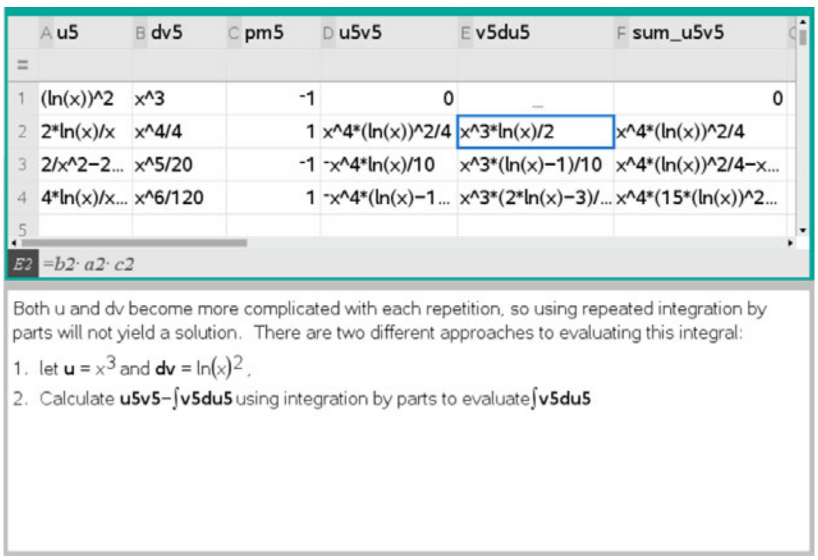


Figure 17: Evaluation of $\int \arctan(x)dx$ with the DI Method

Example 8.0.5. *Evaluate* $\int x^3 \ln(x)^2 dx$.

Discussion:

Choose $\ln(x)^2$ as u and x^3 as dv and configure a TI-Nspire spreadsheet. After performing several iterations of the integration process, both du and vdu become more complex and the iteration fails to produce a value for the integral. Figure 18 shows the result of the process. Examining the results reveals that $v5du5$ (in row 2 of the spreadsheet) is an integrable product. There are two approaches to evaluating $\int x^3 \ln(x)^2 dx$: evaluate the integral $\int x^3 \ln(\frac{x}{2})dx$ and subtract the result from $\frac{x^4 \ln(x)^2}{4}$ (the value of $u5v5$ in row 2), or try evaluating the integral with $u = x^3$ and $dv = \ln(x)^2$. Figure 19 shows the result of the first approach and Figure 20 shows the result of the second approach.



	A u5	B dv5	C pm5	D u5v5	E v5du5	F sum_u5v5
=						
1	(ln(x))^2	x^3	-1	0	_	0
2	2*ln(x)/x	x^4/4	1	x^4*(ln(x))^2/4	x^3*ln(x)/2	x^4*(ln(x))^2/4
3	2/x^2−2...	x^5/20	-1	-x^4*ln(x)/10	x^3*(ln(x)−1)/10	x^4*(ln(x))^2/4−x...
4	4*ln(x)/x...	x^6/120	1	-x^4*(ln(x)−1...	x^3*(2*ln(x)−3)/...	x^4*(15*(ln(x))^2...
5						

E2 =b2· a2· c2

Both u and dv become more complicated with each repetition, so using repeated integration by parts will not yield a solution. There are two different approaches to evaluating this integral:

1. let **u** = x^3 and **dv** = $\ln(x)^2$.
2. Calculate **u5v5**−∫**v5du5** using integration by parts to evaluate∫**v5du5**

Figure 18: Evaluation of $\int x^3 \ln(x)^2 dx$ with the DI Method

Example 8.0.6. *Evaluate* $\int x^3 \ln(\frac{x}{2})dx$.

Discussion:

Continuation of Example 8.0.5. Choose $\ln(\frac{x}{2})$ as u and x^3 as dv and configure a TI-Nspire spreadsheet. Evaluate the integral $\int x^3 \ln(\frac{x}{2})dx$ and subtract the result from $\frac{x^4 \ln(x)^2}{4}$ (the value of $u5v5$ in row 2 of Figure 18). Figure 19 shows the result.

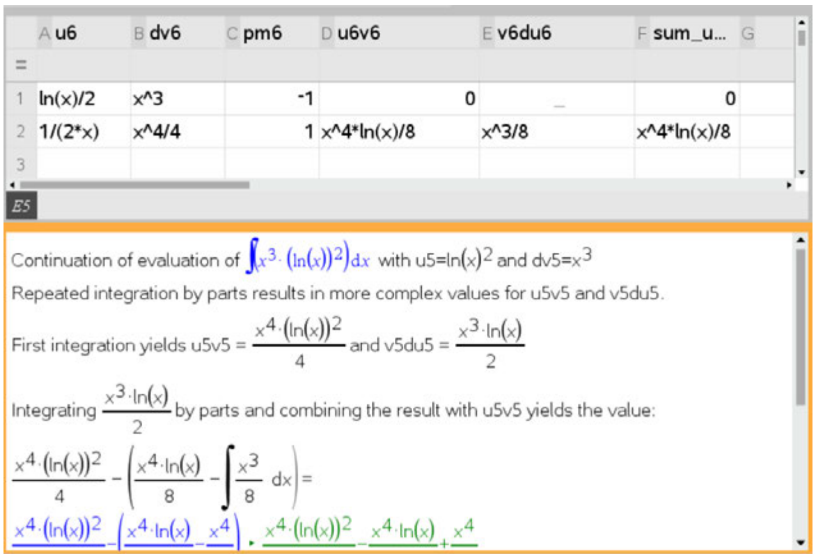


Figure 19: Evaluation of $\int x^3 \ln(\frac{x}{2})dx$ with the DI Method

Example 8.0.7. *Evaluate* $\int x^3 \ln(x)^2 dx$.

Discussion:

Alternative approach to evaluating $\int x^3 \ln(x)^2 dx$ (Example 8.0.5). Choose x^3 as u and $\ln(x)^2$ as dv and configure a TI-Nspire spreadsheet. Copy spreadsheet cells until du, the derivative of u equals zero. The value of the antiderivative without the constant of integration is then the sum of the products of u and v. Figure 20 shows the result.

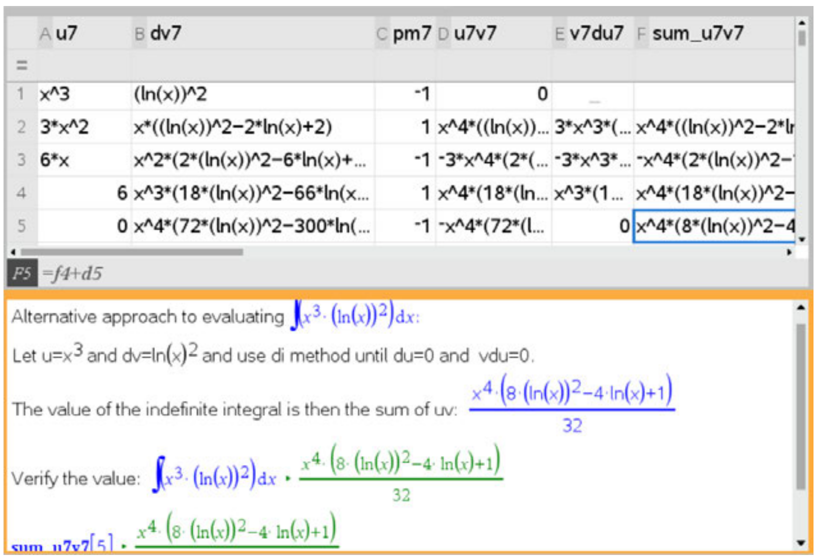


Figure 20: Alternative Evaluation of $\int x^3 \ln(x)^2 dx$ with the DI Method

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